Number Theory for Algebra

Exponents

Rational Numbers

Signed Number Operations

Ratio and Proportion

Variables and Expressions

Equations and Formulas

Data and Probability

Proportional Reasoning

Patterns

The Coordinate Plane

Inequalities

Aiming at Algebraic Intervention

Not business as usual.

Presented by Sarah Cremer
scremer@wested.org

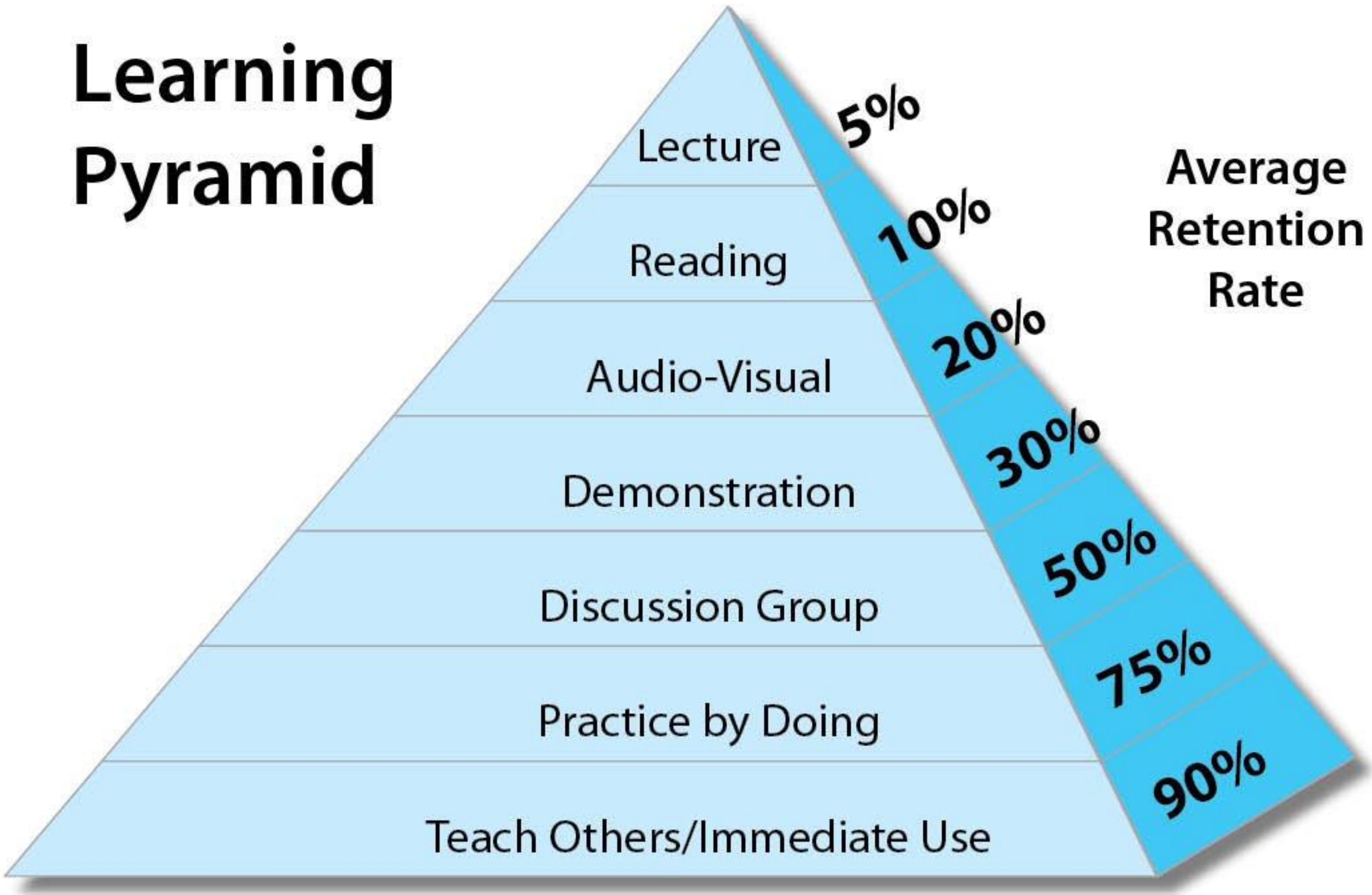
Created by Mardi Gale
mgale@wested.org

"Frank was never any good at arithmetic. He left a third of his money to me, a third to you, a third to Cindy, and a third to Matt."

High achieving students spend more time on concepts and applications, and they cover more topics.

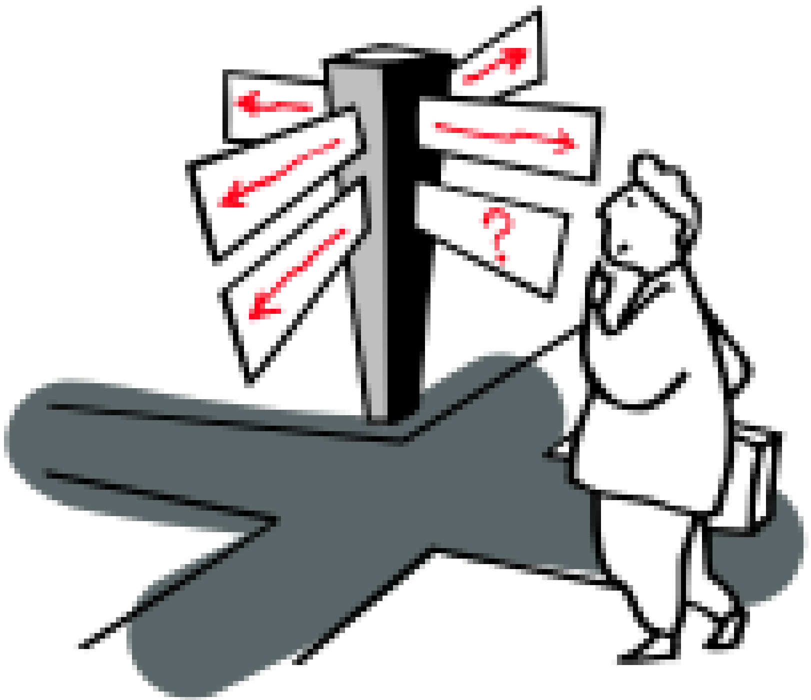
Students working below grade level receive curriculum heavy on review and drill, with a minimum amount of new content.

Learning Pyramid



National Training Laboratories, Bethel, Maine

Which way?



Essential Elements

- Targeted, not comprehensive CURRICULUM
- CONCEPTUALLY based and standards ALIGNED
- FLEXIBLE format and structure
- Embedded INSTRUCTION
- Purposeful task DEVELOPMENT
- Precise, academic LANGUAGE and concrete MODELS
- ASSESSMENT, pre/post and embedded
- Facilitator guides for instructor SUPPORT

Essential Elements

- Targeted, not comprehensive CURRICULUM

Modules

- SignedNumberOperations
- Variables and Expressions
- Ratio and Proportion
- Patterns
- The Coordinate Plane
- Inequalities

Available Now

- NumberTheoryforAlgebra
- Exponents
- Rational Numbers
- Equations and Formulas
- Proportional Reasoning
- Data and Probability

Available Soon

Essential Elements

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- CONCEPTUALLY based and standards ALIGNED

$28 \div 7 = 13$

```
   13
7 ⟌28
  - 7
  ---
   21
 - 21
 ----
```

Check your answer

```
   13
 ×  7
 ----
   21
    7
 ----
   28
```

Check #2:

```
   13
   13
   13
   13
   13
   13
 + 13
 ----
   21
 +  7
 ----
   28
```

47 + 29

29 + 1 = 30
+ 47
―――
77
− 1
―――
76

47 + 3 = 50
29 + 1 = 30
―――
80
− 4
―――
76

40 + 20 = 60
7 + 9 = 16
―――
76

47 + 29

$$
\begin{array}{r}
{\scriptstyle 10} \\
47 \\
+\ 29 \\
\hline
6
\end{array}
\qquad\qquad
\begin{array}{rcl}
40 & + & 7 \\
20 & + & 9 \\
\hline
60 & + & 16
\end{array}
$$

$$47 - 29$$

$$\begin{array}{r} {}^{3}\ \ {}^{1} \\ \cancel{4}7 \\ -\ 29 \\ \hline 18 \end{array}$$

$$\begin{array}{r} 40 + 7 = 47 \\ -\,(20 + 9 = 29) \\ \hline 20 - 2 = 18 \end{array}$$

$$47 \times 29$$

	40	+ 7
20	800	140
+		
9	360	63

$$\begin{array}{r} 40 + 7 \\ \times\ (20 + 9) \\ \hline 800 + 63 \\ 360 + 140 \end{array}$$

$$(40 + 7)(20 + 9) = 800 + 360 + 140 + 63$$

$$(x + 3)\ (x + 5)$$

Standards

	SIGNED NUMBER OPERATIONS	NUMBER THEORY FOR ALGEBRA	VARIABLES AND EXPRESSIONS	RATIONAL NUMBERS	EXPONENTS	EQUATIONS AND FORMULAS	RATIO AND PROPORTION	PATTERNS	THE COORDINATE PLANE	PROPORTIONAL REASONING	INEQUALITIES	DATA AND PROBABILITY
ALGEBRA												
Seeing Structure in Expressions												
Interpret the structure of expressions.			**X**			**X**		X	X			
Write expressions in equivalent forms to solve problems.			**X**	X	X	**X**	X	X	X	X	X	
Arithmetic with Polynomials and Rational Expressions												
Perform arithmetic operations on polynomials.	X		X	X		X	X	X	X	X	X	
Understand the relationship between zeroes and factors of polynomials.												
Use polynomial identities to solve problems.												
Rewrite rational expressions.		X	X	**X**	X		X			X		
Creating Equations												
Create equations that describe numbers or relationships.						**X**	**X**	**X**	**X**	X	X	
Reasoning with Equations and Inequalities												
Understand solving equations as a process of reasoning and explain the reasoning.						**X**		X	X	X		
Solve equations and inequalities in one variable.						**X**		X	**X**	X	**X**	
Solve systems of equations.									**X**			
Represent and solve equations and inequalities graphically.									**X**		**X**	

Note: an X indicates the standard is addressed in that *Aim for Algebra* module. A bold **X** indicates the standard is a primary focus of the module.

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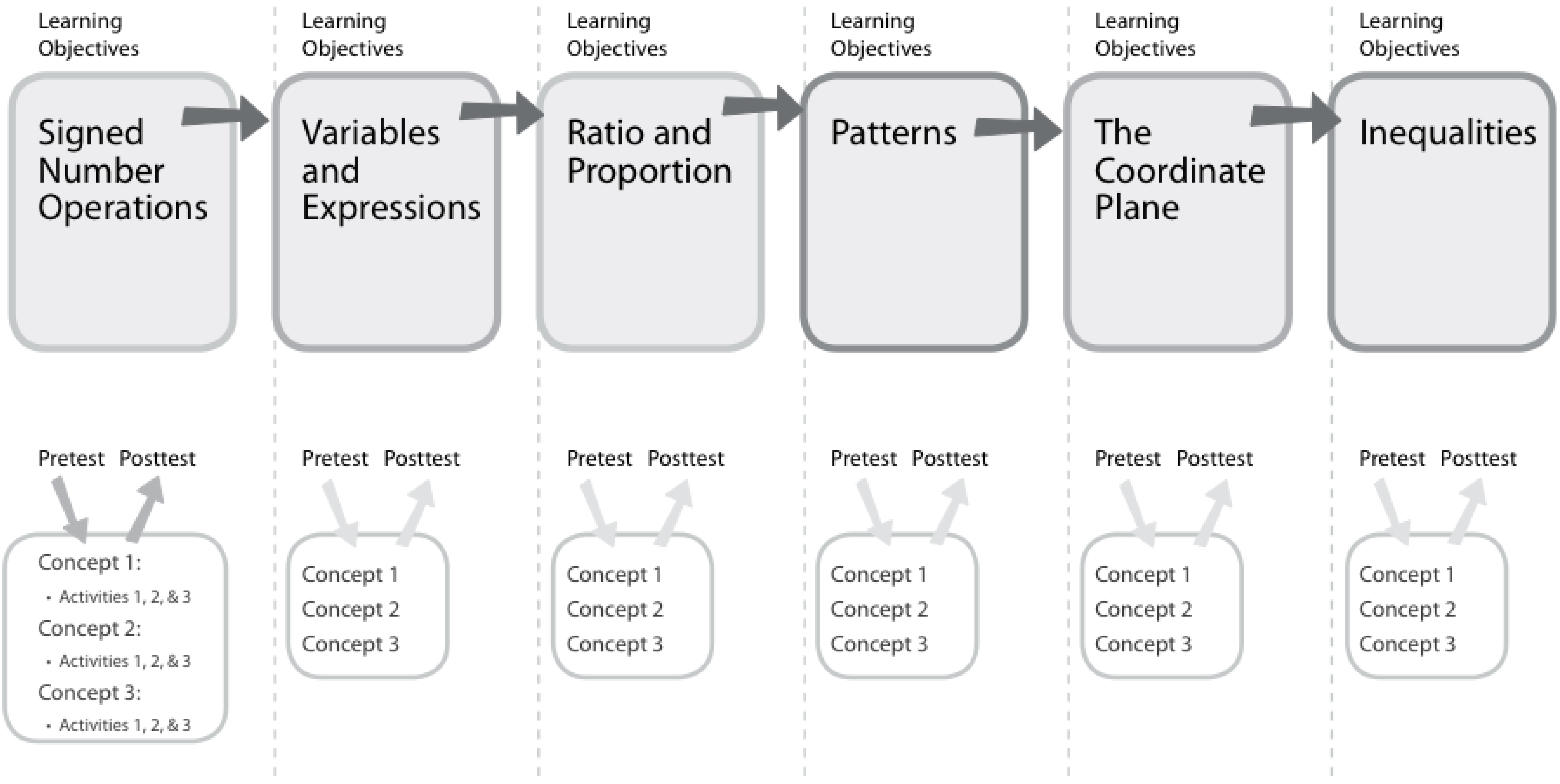
Flexibility and Fidelity



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- Embedded INSTRUCTION

More than One Interpretation

Sometimes the language used to ask a question can be interpreted in different ways.

1. One algebra class has 20 students and a second class has 25 students.
   The first class has 10 girls, and the second class has 12 girls.
   Which class has more girls?

   There are at least two ways to answer this question.

   a. The second class has more girls because:

   12 is greater than 10.

   This interpretation compares the **parts** (the number of girls in each class).

   b. The first class has more girls because:

   The first class has 10 girls and 20 students ($\frac{10}{20}$).

   The second class has 12 girls and 25 students ($\frac{12}{25}$).

   $\frac{10}{20}$ has greater value than $\frac{12}{25}$.

   $\frac{12}{25}$ is less than half.

   This interpretation compares the **parts** (the number of girls in each class) to the **whole** (the total number of students in each class).

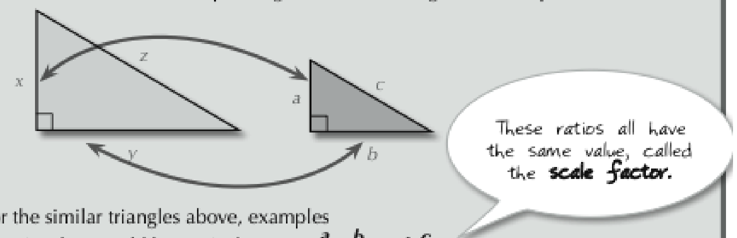
   Both interpretations are correct when justified mathematically.

2. An English class of 20 students is all girls.
   A second class has 25 students, 22 of whom are girls.

   Which class has more girls? Answer this question two ways.

Other Equivalent Ratios for Similar Figures

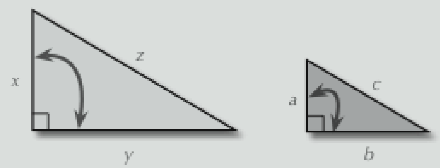
You've seen that ratios of corresponding sides of similar figures have equivalent ratios.



For the similar triangles above, examples of ratios that would be equivalent are: $\frac{a}{x}$, $\frac{b}{y}$, **and** $\frac{c}{z}$.

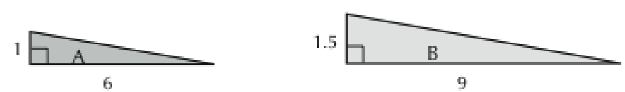
◆ ◆ ◆

Similar figures have other ratios that are equivalent as well.



These ratios will also be equivalent: $\frac{x}{y}$ **and** $\frac{a}{b}$, $\frac{z}{y}$ **and** $\frac{c}{b}$, $\frac{x}{z}$ **and** $\frac{a}{c}$.

The right triangles A and B below are **similar**.



Use the right triangles to complete the following:

1. Find the $\frac{base}{height}$ ratio for Triangle A. __________________
2. Find the $\frac{base}{height}$ ratio for Triangle B. __________________
3. Are these ratios equivalent? __________________

AFTERNOON SNACKS

Jesse and Aaron each bought the same items at the snack store.

Two snack bars

Four fruit drinks

Aaron wrote down the following expression to figure out how much it would cost each of them.

$$2 \cdot 3 + 4 \cdot 2$$

2 snack bars at $3.00 each and 4 fruit drinks at $2.00 each.

Jesse thought the total cost should be $20.00.		Aaron thought the total cost should be $14.00.	
	$2 \cdot 3 + 4 \cdot 2$		$2 \cdot 3 + 4 \cdot 2$
First Jesse multiplied:	$= 6 + 4 \cdot 2$	First Aaron multiplied:	$= 6 + 8$
Then he added:	$= 10 \cdot 2$	Then he added:	$= 14$
Then he multiplied:	$= 20$		

1. Who was correct? Explain why his answer was correct.

2. What error did the other person make?

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How does the number of tables relate to the number of addends of 3?

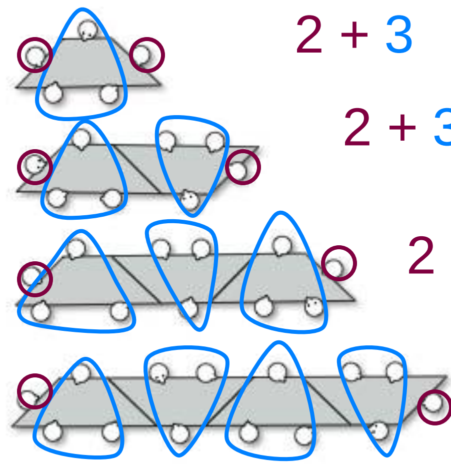
$2 + 3t$

Number of tables

Seats at the Table

Mario wants to know how many people he can seat around a row of trapezoid shaped tables.

He started by drawing the sketches below. Then he made a chart.



Help Mario by answering the following questions.
Use what you've learned about sequences and the recursive rule.

1. Complete Mario's chart:

Number of tables in a row:	1	2	3	4	5	6	7	8
Number of people seated:	5							

Think of the 'Number of people seated' as its own numerical sequence.

2. Mario says that he can seat 42 people around 13 tables. Is he correct? ____________

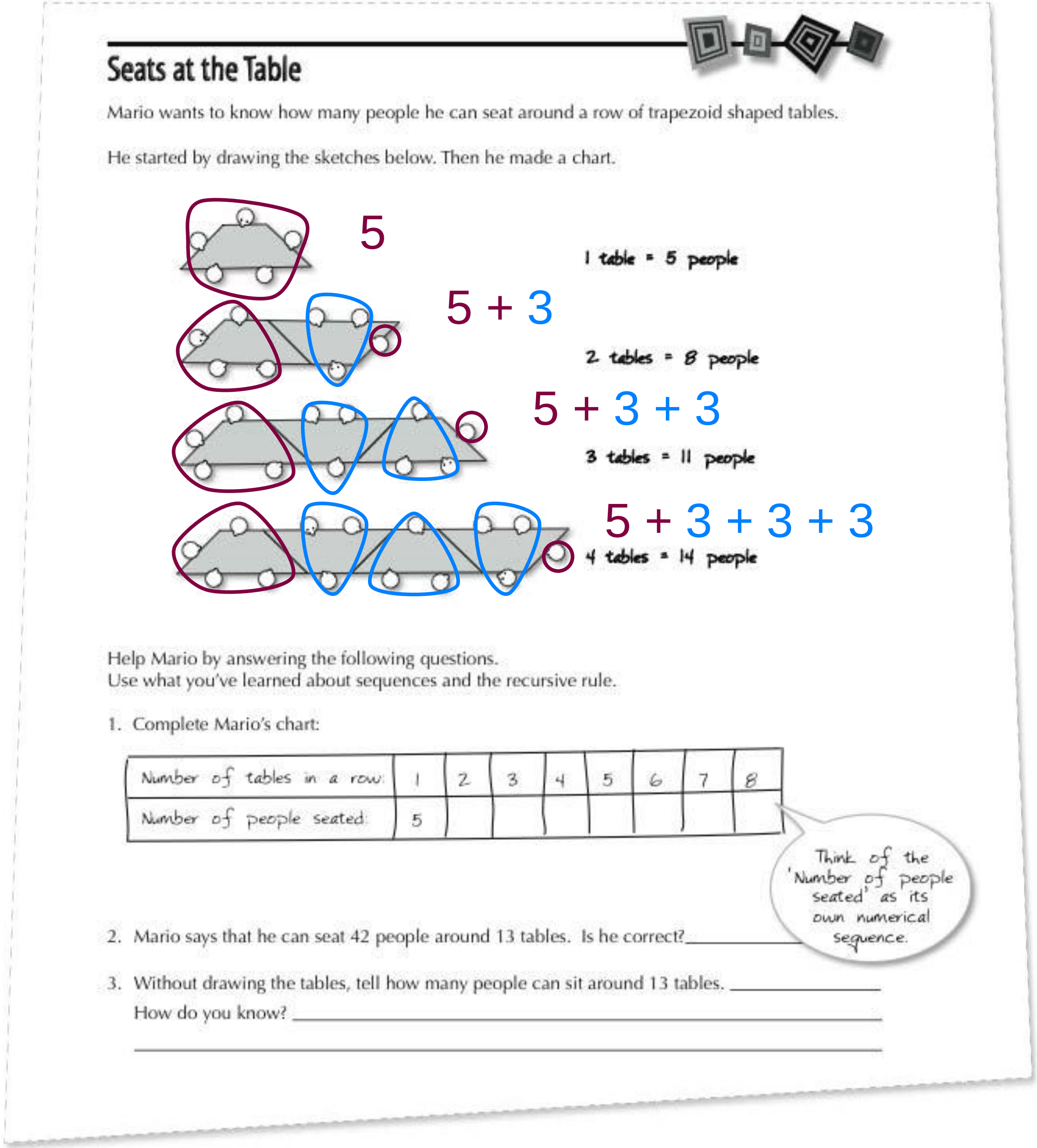
3. Without drawing the tables, tell how many people can sit around 13 tables. ____________
How do you know? ____

____

Now how does the number of tables relate to the number of addends of 3?

$5 + 3(t - 1)$

Number of tables less 1



Seats at the Table

Mario wants to know how many people he can seat around a row of trapezoid shaped tables.

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Help Mario by answering the following questions.
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Think of the 'Number of people seated' as its own numerical sequence.

2. Mario says that he can seat 42 people around 13 tables. Is he correct? ______

3. Without drawing the tables, tell how many people can sit around 13 tables. ______
How do you know? ______

Now how does the number of tables relate to the number of addends of 3?

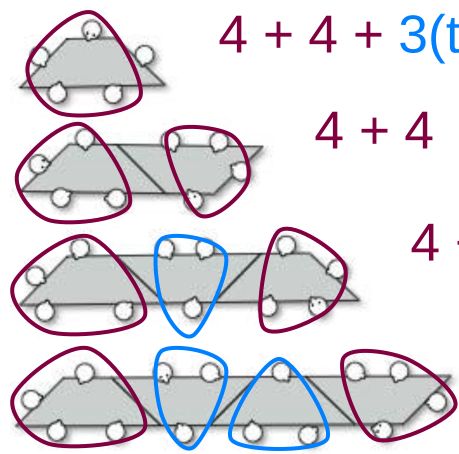
$4 + 4 + 3(t - 2)$

Number of tables less 2

Seats at the Table

Mario wants to know how many people he can seat around a row of trapezoid shaped tables.

He started by drawing the sketches below. Then he made a chart.



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Think of the 'Number of people seated' as its own numerical sequence.

2. Mario says that he can seat 42 people around 13 tables. Is he correct?____________

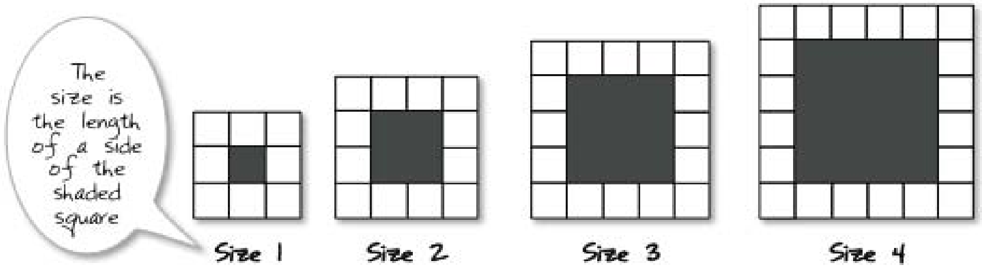
3. Without drawing the tables, tell how many people can sit around 13 tables. ____________
How do you know? ____________________________________

Different ways students describe how the pattern grows:

$2 + 3t$	$4 + 4 + 3(t - 2)$
$3t + 2$	$8 + 3(t - 2)$
$5 + 3(t - 1)$	$5t - 2(t - 1)$

Reconcile

BORDER TILES



Bob drew the patterns above. Small white tiles surround shaded squares.
The shaded squares have different side lengths. Then he made the table on the right.

1. Fill in the empty boxes in Bob's table.
2. Bob thinks that he will need a total of 50 tiles to surround a shaded square with side length of 12 inches. Is Bob correct? How do you know?
3. How many small white squares are needed to surround a shaded square with side length of 12?

   ____

4. Write the rule describing how you determined the correct number of squares for size 12, in words **and** in algebraic notation.
   Refer to the geometric figures in your explanation of your rule.

Length of Side of Shaded Square (*s*)	**Number of Small White Squares (*w*)**
1	8
2	12
3	16
4	
5	
6	
7	

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Which is the largest number?

4

-7

A Symbol With Different Meanings

The "–" symbol can take on different meanings in mathematics depending on how it is used. Fill in the chart below with one of your own examples, and then describe each expression in words.

The "–" symbol may tell you a number is less than zero.
So the number that follows the symbol is **negative**:

symbols	**examples of what we say**
-7	
-0.5	
(choose your own example)	

The "–" symbol may mean the **opposite** of the quantity that follows:

$-(y + 2)$	
$-x$	
$-(-4)$	

The "–" symbol may mean the operation of **subtraction**:

$11 - 8$	
$x - 3$	

Express Your Age

1. What is your age now? ______
2. How old will you be two years from now? ______
3. How old will you be ten years from now? ______
4. What is twice your age? ______
5. What is half your age? ______
6. In how many years will you be 50? ______

Nora and Max

What if we don't know someone's age and want to ask some questions?

Let ***n*** represent Nora's present age in years.

EXAMPLE

1. What is Nora's age now? n
2. How old will Nora be two years from now? $n + 2$
3. How old will she be ten years from now? ______
4. What is twice Nora's current age? ______
5. What is half Nora's age? ______
6. How old will Nora be 50 years from now? ______
7. In how many years will she be 50 years old? ______

THINK
How are questions 6 and 7 different?

Let ***m*** represent Max's present age in years.

EXAMPLE

8. What is the sum of Nora's and Max's ages now? $n + m$
9. What is the sum of Nora's and Max's ages ten years from now? ______

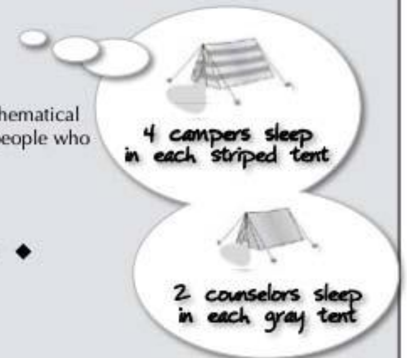
A Mathematical Model

An algebraic expression which represents a real world situation is a **"mathematical model."** A mathematical model can help show how a situation changes for different values of the variables.

Here is an example of a mathematical model from Camp Splash.

Let: $\boldsymbol{t}$ = number of 2-person tents.

$\boldsymbol{f}$ = number of 4-person tents



The algebraic expression $\mathbf{2t + 4f}$ is a mathematical model that represents the total number of people who can sleep in the tents.

◆ ◆ ◆

Fill in the blanks for the following questions about the expression $2t + 4f$.

1. The first term 2t represents the total number of counselors in the 2-person tents.

2. The second term ____ represents ________________________________.

3. The numerical coefficient of the first term 2, indicates there are 2 counselors per 2-person tent.

4. The numerical coefficient of the second term, ______, indicates ________________________________.

Why it Works

We can use algebra to show why this number game on page 1 works for any number you choose.

Let the variable x represent any number.

Fill in the blanks below

Steps	Algebraic Notation
1. Pick any number	x
2. Subtract 2	$x - \mathbf{2}$
3. Multiply by 3	$\mathbf{3}(x - 2)$
Simplify using the Distributive Property	= ________________
4. Add 15	$3x - 6 \mathbf{+ 15}$
Simplify by combining like terms	= ________________
5. Divide by 3	$\frac{3x + 9}{\mathbf{3}}$
Simplify using the inverse of the Distributive Property	$= \frac{3(x + 3)}{3}$ $= x + 3$
6. Subtract the original number	$(x + 3) - \boldsymbol{x}$
Simplify	= ________________

What is your final answer?

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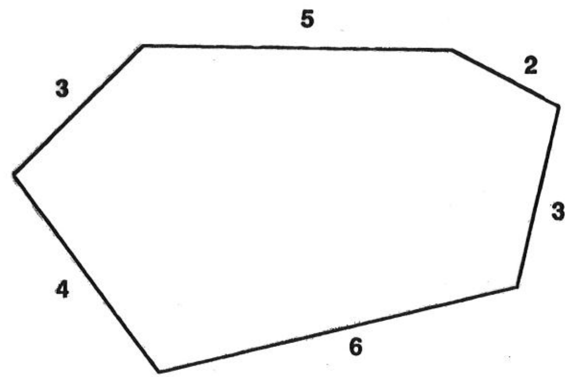
◆ **A typical geometry test question today:**

Find the perimeter.



It is possible to answer this question correctly without having any idea what a perimeter is.

- A problem that gives a better indication of a student's knowledge of perimeter is:

Draw a six-sided irregular polygon with a perimeter of 23 units. Show all dimensions.

Sources of Assessment Items

- Linked directly to instruction
- Includes items from national exams, such as NAEP and TIMSS
- Includes items adapted from state standard exams and state exit exams

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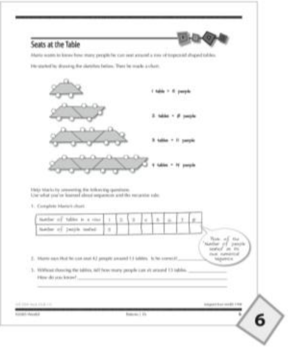
PURPOSE: Apply understanding of recursive rules and patterns using geometric representations.

This task may serve as a formative assessment.



LAUNCH: Before giving students the item to work on:

- Ask students if they can think of situations where they might need to arrange tables to maximize seating. For example, party planning, planning for family meals at Thanksgiving or during the holiday season, or soccer banquets. Discuss how they would decide the table arrangement.
- Draw the first picture on the overhead/board or use pattern blocks to show the concrete model of one table.
- Establish the way in which people sit at the tables. That is, 1 person on each end, 1 on the short side, and 2 on the long side.
- Then model putting a second table on the overhead, or show or sketch the second picture from the task.
- Discuss where people can sit now that the tables have been pushed together. Remind them that the rule for adding tables is that they connect end to end, as shown.
- Read the directions together to be sure students understand the questions asked before students complete the items.
- Students may work on the items independently.

A great set up for the situation: If available, read the book *Spaghetti And Meatballs For All (Marilyn Burns Brainy Day Books)* by Marilyn Burns and Gordon Suilveria to the class. It describes a family gathering where more and more people are invited and the table arrangement needs to grow in order to accommodate the number of people.

SUMMARY:

- Ask students to share their responses. Accept many different explanations for item 3.
- Ask students to use the diagram to explain their solution. That is, explain their responses in terms of the picture. (When another table is added, two of the ends are no longer useable, but the person on one end is now sitting at the end of the new table and each time a table is added, 3 more people can sit with the group – circle the 3 people.)
- Ask students to make the connection between the diagram, the column at the right of the diagram, and the table in item 1. (They all represent the same pattern, but are displayed in different formats.)

  This will serve as a reflection and analysis on the work they have performed.

Express Your Age, Purpose: To provide practice creating a numerical example of an expression to describe a quantity.

Launch, Top of Page:

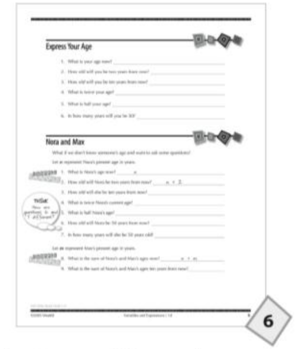
- Remind students the expression itself represents the quantity described, so no equation or variable to represent the answer is necessary.
  For example: Items 5 and 6 can be represented in several different ways and all of them represent the (single) quantity equal to half the age given:
  $\text{age} \div 2; \frac{1}{2}\text{age}; \frac{\text{age}}{2}$
- Recognizing that the expression represents the quantity asked for is important for items 6 and 7 in "Nora and Max." See *Summary* below.
- Students will have different responses for these items. As they share their responses, ask them to share how they found the quantities. For example, if their age is 14, how did they find the answer to item 4? Ask students to represent their process with numerals: 14(2) or 2(14).

Nora and Max, Purpose: To extend the practice of representing information or data with numbers to representing quantities with numbers and variables, thereby creating algebraic expressions.

Launch, Bottom of Page:

- Ask students to consider that Nora may not want anyone to know how old she is so Nora is using a variable (n) to represent her age in the expressions they are about to create.
- Remind students to use appropriate grouping symbols where necessary to show a quantity, as in item 2. This is read "the quantity of n plus two." The parentheses are optional in this expression but are helpful when reading the expressions. See *Summary*.
- Since there are no equal signs in algebraic expressions, there are none necessary for this page. The expression represents the quantity described.

Summary: It is important that students are familiar with and use the phrase "the quantity of…," as indicated by grouping symbols, when reading expressions. Recognizing the expression as one quantity helps students learn how and when to use the properties of operation to simplify and/or evaluate expressions, particularly the Distributive Property. As students share their responses, be sure they use correct language in reading their expressions.



Possible Student Responses/Misconceptions

Students need to recognize that it is the expression itself that represents the answer to the question; no equation or variable is necessary.

For example: "$\frac{\text{age}}{2}$" is the expression that represents half your age; "50 – (age)" is the expression that represents how many years until you are fifty.

Key Mathematical Idea

Students may try to create equations for items 6 and 7, such as $n + 50 = x$ or $50 = n + x$. These are not expressions and introduce a second variable, both of which are not expected or necessary for this task.

The expression "$n + 50$" represents Nora's age in 50 years in and of itself; no other notation is necessary.

In the same way, "$50 - n$" represents the number of years until Nora is 50 years old. Another way to think about this expression is to say, "the quantity $50 - n$ is the number of years until Nora is 50 years old."

Presented by
Sarah Cremer
scremer@wested.org

Created by Mardi Gale
mgale@wested.org

For more information:

Kimberly Viviani
kviviani@wested.org

Its About Time
www.its-about-time.com/aim/aim.html

Webinar
http://schoolsmovingup.net/webinars/algebra

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